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Front Form Spinors in Weinberg-Soper Formalism and Melosh Transformations for any Spin*

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Abstract

Using the Weinberg-Soper formalism we construct the front form $(j, 0) \oplus (0, j)$ spinors. Explicit expressions for the generalised Melosh transformations up to spin two are obtained. The formalism, without explicitly invoking any wave equations, reproduces spin one half front-form results of Melosh, Lepage and Brodsky, and Dziembowski.

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I. INTRODUCTION

Following the work of Weinberg [1] and Soper [2] we extend our recent work [3] and explicitly construct the front form covariant spinors for arbitrary spin j *without* a direct reference to any wave equation. We also obtain a generalisation of Melosh transformation, $\Omega(j)$, to $(j, 0) \oplus (0, j)$ fields. Explicit examples for $j = \frac{1}{2}, 1, \frac{3}{2}, 2$ are given. For $j = \frac{1}{2}$, the formalism reproduces the front form spinors of Lepage and Brodsky [4]; and $\Omega(\frac{1}{2})$ coincides with the celebrated “Melosh transformation” [5,6]. This approach is in the spirit of Dirac’s [7] original motivation for the high-spin wave equations for “approximate application to composite particles” and opens up the possibility of constructing a QCD based effective field theory of hadronic resonances along the lines of recent work of Cahill [8].

Unless otherwise indicated we follow the notation of Refs. [9, 3(b)]. We use the notation $x^\mu = (x^+, x^1, x^2, x^-)$. In terms of the instant form variables $\underline{x}^\mu = (x^\circ, x^1, x^2, x^3)$, we have $x^\pm = x^\circ \pm x^3$. The evolution of a system is studied along the coordinate x^+ , and as such it plays the role of “time.”

II. FRONT FORM HADRONIC SPINORS

We will work under the assumption that the center of mass of a composite hadrons is best described, in the phenomenological sense, by the fields constructed from the $(j, 0) \oplus (0, j)$ spinors in the front form. Following the Weinberg-Soper formalism [1,2], these spinors will be constructed from the right handed, $\phi^R(p^\mu)$, and left handed, $\phi^L(p^\mu)$, matter fields. We begin from the transformation which takes a particle from rest, $\hat{p}^\mu = (m, 0, 0, m)$, to a particle moving with an arbitrary four momentum $p^\mu = (p^+, p^1, p^2, p^-)$ [Note: for massive particles $p^+ > 0$].

In the *instant form of field theory* the transformation which takes the rest momentum $\hat{p}^\mu = (m, 0, 0, 0) \rightarrow \underline{p}^\mu = (p^\circ, \vec{p})$, is constructed out of the boost operator \vec{K} and is given by

$$\underline{p}^\mu = \Lambda^\mu{}_\nu \hat{p}^\nu, \quad (1)$$

with

$$\Lambda = \exp(i\vec{\varphi} \cdot \vec{K}). \quad (2)$$

In Eq. (2) the boost parameter $\vec{\varphi}$ is defined as follows

$$\hat{\varphi} = \vec{p}/|\vec{p}|, \quad \cosh(\varphi) = E/m, \quad \sinh(\varphi) = |\vec{p}|/m. \quad (3)$$

Note that the stability group of the $x^\circ = 0$ plane consists of the six generators \vec{J} and \vec{P} . The \vec{K} , along with the P° , generate the instant-form dynamics.

In the *front form of field theory* the transformation which takes $\hat{p}^\mu = (m, 0, 0, 0) \rightarrow \underline{p}^\mu = (p^\circ, \vec{p})$ is defined [2,10] by

$$\underline{p}^\mu = L^\mu{}_\nu \hat{p}^\nu, \quad (4)$$

with the matrix L given by

$$L = \exp(i\vec{v}_\perp \cdot \vec{G}_\perp) \exp(i\eta K_3). \quad (5)$$

The parameters η and $\vec{v}_\perp = (v_x, v_y)$ specify a given boost. The generators \vec{G}_\perp are defined as follows

$$G_1 = K_1 - J_2, \quad G_2 = K_2 + J_1, \quad (6)$$

and together with

$$P_- = P_0 - P_3, \quad P_1, \quad P_2, \quad J_3, \quad K_3, \quad (7)$$

form the seven generators of the stability group of the $x^+ = 0$ plane. (Note that $P_- = P^+$). The algebra associated with the stability group is summarised in Table I. The generators $D_1 = K_1 + J_2$, $D_2 = K_2 - J_1$ and $P_+ = P_0 + P_3$ generate the front-form dynamics.

It is important to note that while the front-form transformation L is specified entirely in terms of the generators of the $x^+ = 0$ plane stability group, the instant-form transformation Λ involves dynamical generators associated with the $x^0 = 0$ plane.

Using the explicit matrix expressions for $\vec{J} = (J_1, J_2, J_3)$ and $\vec{K} = (K_1, K_2, K_3)$ given in Eqs. (2.65 - 2.67) of Ref. [9] we obtain an explicit expression for the boost L defined in Eq. (5)

$$[L^\mu{}_\nu] = \begin{bmatrix} \cosh(\eta) + \frac{1}{2}\vec{v}_\perp^2 \exp(\eta) & v_x & v_y & \sinh(\eta) + \frac{1}{2}\vec{v}_\perp^2 \exp(\eta) \\ v_x \exp(\eta) & 1 & 0 & v_x \exp(\eta) \\ v_y \exp(\eta) & 0 & 1 & v_y \exp(\eta) \\ \sinh(\eta) - \frac{1}{2}\vec{v}_\perp^2 \exp(\eta) & v_x & v_y & \cosh(\eta) - \frac{1}{2}\vec{v}_\perp^2 \exp(\eta) \end{bmatrix}. \quad (8)$$

Recalling that the components of the front form momentum p^μ are defined as $p^\pm = p^0 \pm p^3$ this yields

$$p^+ = m \exp(\eta), \quad \vec{p}_\perp = m \exp(\eta) \vec{v}_\perp, \quad p^- = m \exp(-\eta) + m \exp(\eta) \vec{v}_\perp^2. \quad (9)$$

The variables η and \vec{v}_\perp are fixed by requiring \underline{p}^μ generated by Eq. (1) to be identical to the \underline{p}^μ produced by Eq. (4).

Given the transformation L , Eq. (5), we now wish to construct the front form $(j, 0) \oplus (0, j)$ hadronic spinors. To proceed in this direction we rewrite L by expanding the exponentials in Eq. (5), and using Table I to arrive at [11]

$$L = \exp \left[i \left(a \vec{v}_\perp \cdot \vec{G}_\perp + \eta K_3 \right) \right], \quad (10)$$

with

$$a = \frac{\eta}{1 - \exp(-\eta)}. \quad (11)$$

For the $(j, 0)$ matter fields, $\phi^R(p^\mu)$, we have [1,9,3(g)] $\vec{K} = -i \vec{J}$. For the $(0, j)$ matter fields, $\phi^L(p^\mu)$, $\vec{K} = +i \vec{J}$. Using this observation, along with Eq. (10) and definitions (6), we obtain the transformation properties of the front form $(j, 0)$ and $(0, j)$ hadronic fields

$$\phi^R(p^\mu) = \exp \left(+\eta \hat{b} \cdot \vec{J} \right) \phi^R(\hat{p}^\mu), \quad (12)$$

and

$$\phi^L(p^\mu) = \exp\left(-\eta \hat{b}^* \cdot \vec{J}\right) \phi^L(\hat{p}^\mu), \quad (13)$$

where the unit vectors \hat{b} and \hat{b}^* are given by

$$\hat{b} = \eta^{-1} (a v_r, -i a v_r, \eta), \quad (14)$$

$$\hat{b}^* = \eta^{-1} (a v_\ell, i a v_\ell, \eta); \quad \hat{b} \cdot \hat{b} = 1 = \hat{b}^* \cdot \hat{b}^*, \quad (15)$$

with $v_r = v_x + i v_y$ and $v_\ell = v_x - i v_y$.

We now make two observations. First, under the operation of *parity* we have $\phi^R(p^\mu) \leftrightarrow \phi^L(p^\mu)$. Second, for the particle at rest ($\vec{p} = \vec{0}$, $p^\mu = \hat{p}^\mu$) the concept of handedness loses its physical significance, (c.f. Ref. [9] and more detailed discussion in Ref. [12]), and this in turn yields the relation

$$\phi^R(\hat{p}^\mu) = \pm \phi^L(\hat{p}^\mu). \quad (16)$$

We now introduce the spin- j hadronic spinor

$$\psi(p^\mu) = \frac{1}{\sqrt{2}} \begin{bmatrix} \phi^R(p^\mu) + \phi^L(p^\mu) \\ \phi^R(p^\mu) - \phi^L(p^\mu) \end{bmatrix}, \quad (17)$$

and observe that the plus (minus) sign in Eq. (16) yields spinors with *even (odd) spinor parity*. We will denote the even spinor parity spinors by $\mathcal{U}(p^\mu)$; and the odd spinor parity spinors by $\mathcal{V}(p^\mu)$.

The transformation property for these hadronic spinors under the boost (5) is now readily obtained by using Eqs. (12) and (13). The result is

$$\psi(p^\mu) = M(L) \psi(\hat{p}^\mu), \quad (18)$$

with the operator $M(L)$ given by

$$M(L) = \frac{1}{2} \begin{bmatrix} \exp(\eta \hat{b} \cdot \vec{J}) + \exp(-\eta \hat{b}^* \cdot \vec{J}) & \exp(\eta \hat{b} \cdot \vec{J}) - \exp(-\eta \hat{b}^* \cdot \vec{J}) \\ \exp(\eta \hat{b} \cdot \vec{J}) - \exp(-\eta \hat{b}^* \cdot \vec{J}) & \exp(\eta \hat{b} \cdot \vec{J}) + \exp(-\eta \hat{b}^* \cdot \vec{J}) \end{bmatrix}. \quad (19)$$

In what follows we present the explicit construction of the hadronic spinors $\mathcal{U}(p^\mu)$ and $\mathcal{V}(p^\mu)$ for $j = \frac{1}{2}, 1, \frac{3}{2}, 2$.

III. CONSTRUCTION OF $\mathcal{U}(p^\mu)$ AND $\mathcal{V}(p^\mu)$ AND THEIR PROPERTIES

Let us first note that the front form helicity operator

$$\mathcal{J}_3 \equiv J_3 + \frac{1}{P_-} (G_1 P_2 - G_2 P_1), \quad (20)$$

introduced by Soper [2] and discussed by Leutwyler and Stern [10], commutes with all generators of the stability group associated with the $x^+ = 0$ plane. The front form helicity operator associated with the $(j, 0) \oplus (0, j)$ spinors constructed above is then readily defined to be

$$\Theta = \begin{bmatrix} \mathcal{J}_3 & 0 \\ 0 & \mathcal{J}_3 \end{bmatrix}. \quad (21)$$

If we choose a matrix representation of the \vec{J} operators with J_3 diagonal (we follow the standard convention of Ref. [13]) then the $2(2j + 1)$ -element basis spinors for a particle at rest have the general form

$$\mathcal{U}_{+j}(\vec{p}^\mu) = \begin{bmatrix} \mathcal{N}(j) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathcal{U}_{j-1}(\vec{p}^\mu) = \begin{bmatrix} 0 \\ \mathcal{N}(j) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathcal{V}_{-j}(\vec{p}^\mu) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \mathcal{N}(j) \end{bmatrix}. \quad (22)$$

The index $h = j, j-1, \dots, -j$ on the $\mathcal{U}_h(p^\mu)$ and $\mathcal{V}_h(p^\mu)$ corresponds to the eigenvalues of the front form helicity operator Θ . The spin-dependent normalisation constant $\mathcal{N}(j)$ is to be so chosen that for the *massless* particles all $\mathcal{U}_h(\vec{p}^\mu)$ and $\mathcal{V}_h(\vec{p}^\mu)$ vanish (There can be no massless particles at rest !), and the only non-vanishing spinors are $\mathcal{U}_{h=\pm j}(p^\mu)$ and $\mathcal{V}_{h=\pm j}(p^\mu)$. The simplest choice satisfying these requirements is

$$\mathcal{N}(j) = m^j. \quad (23)$$

We now have all the details needed to construct $\mathcal{U}_h(p^\mu)$ and $\mathcal{V}_h(p^\mu)$ for any hadronic field. In this paragraph we summarise the algebraic construction used for $j = \frac{1}{2}, 1, \frac{3}{2}, 2$. Using Eqs. (A27,A28) and (A31,A32) of Ref. [1] along with Eqs. (9,14,15) above we obtain the expansions for the $\exp(\eta \hat{b} \cdot \vec{J})$ and $\exp(-\eta \hat{b}^* \cdot \vec{J})$ which appear in the light-front $(j, 0) \oplus (0, j)$ -spinor boost matrix $M(L)$, Eq. (19). These expansions are presented in Appendix A. Using the results of Appendix A, explicit expressions for $M(L)$, Eq. (19), are then calculated as a simple, but somewhat lengthy algebraic exercise. These expressions for $M(L)$ combined with Eqs. (18) and (22) yield the hadronic spinors presented in Appendix B. The generality of the procedure for any spin is now obvious, and the procedure reduces to the well defined algebraic manipulations.

We now introduce the following useful matrices,

$$\Gamma^\circ = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \Gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad (24)$$

with $I = (2j + 1) \times (2j + 1)$ identity matrix.

In reference to the spinors presented in Appendix B, we define

$$\bar{\psi}(p^\mu) \equiv \psi^\dagger(p^\mu) \Gamma^\circ. \quad (25)$$

Using the explicit expressions for $\mathcal{U}(p^\mu)$ and $\mathcal{V}(p^\mu)$, Eqs. (B1-B7), we verify that

$$\bar{\mathcal{U}}_h(p^\mu) \mathcal{U}_{h'}(p^\mu) = m^{2j} \delta_{hh'}, \quad (26)$$

$$\bar{\mathcal{V}}_h(p^\mu) \mathcal{V}_{h'}(p^\mu) = -m^{2j} \delta_{hh'}, \quad (27)$$

$$\bar{\mathcal{U}}_h(p^\mu) \mathcal{V}_{h'}(p^\mu) = 0 = \bar{\mathcal{V}}_h(p^\mu) \mathcal{U}_{h'}(p^\mu). \quad (28)$$

For spin one half the result given by Eq. (B1) corresponds to that given by Lepage and Brodsky [4, Eq.A3]¹. A noteworthy feature of the front-form spinors constructed here in the $(j, 0) \oplus (0, j)$ representation is the observation that:

1. For the massless case, $m = 0$, the $\mathcal{U}_h(p^\mu)$ and $\mathcal{V}_h(p^\mu)$ identically vanish unless the associated front form helicity $h = \pm j$; and
2. For the massive case, $m \neq 0$, an examination of Eqs. (B3), (B4) and (B5-B7) yields the result that in the “high momentum” limit, $p^+ \gg m$, the leading asymptotic behavior of $\mathcal{U}_h(p^\mu)$, and $\mathcal{V}_h(p^\mu)$ is given by $\sim (p^+)^{|h|}$.

The correspondence between observations I and II is to be perhaps fully realised when various matrix elements, of appropriate front-form operators \mathcal{O} , $\bar{\psi}_{h'}(p'^\mu) \mathcal{O} \psi_h(p^\mu)$, are studied for the massive case. Since the high momentum behaviour of $\mathcal{U}_h(p^\mu)$ and $\mathcal{V}_h(p^\mu)$ is $\sim (p^+)^{|h|}$, we can expect to generalise to arbitrary spin the results on helicity amplitudes presented by Lepage and Brodsky [4] for $j = 1/2$. It is expected that the dominant *elastic-scattering* amplitudes will correspond to the helicity non-changing processes, while the helicity changing transitions will be suppressed by appropriate powers of the factor (m/p^+) . The fact that the front form spinors $\mathcal{U}_{|h|<j}(p^\mu)$ and $\mathcal{V}_{|h|<j}(p^\mu)$ for massive particles do *not* identically vanish in the high momentum limit is of profound physical significance. To see this note that, as argued by Brodsky and Lepage [14], the hadrons in $e^+ + e^- \rightarrow \gamma^* \rightarrow \mathcal{H}_A + \overline{\mathcal{H}}_B$ are produced at large Q^2 with opposite helicity $h_A + h_B = 0$ and $|h_i| \leq 1/2$. As a consequence, to give an example [14, Table I], the process $e^+ + e^- \rightarrow p_{\pm\frac{1}{2}} + \overline{\Delta}_{\pm\frac{3}{2}}$ is *suppressed* relative to $e^+ + e^- \rightarrow p_{\pm\frac{1}{2}} + \overline{\Delta}_{\mp\frac{1}{2}}$. Here, the physically dominant degree of freedom is *not* $\Delta_{\pm\frac{3}{2}}$ but $\Delta_{\pm\frac{1}{2}}$ — the degree of freedom which for massless case identically vanishes.

IV. GENERALISED MELOSH TRANSFORMATION – THE CONNECTION WITH THE INSTANT FORM

In a representation appropriate for comparison with the front form spinors $\mathcal{U}_h(p^\mu)$ and $\mathcal{V}_h(p^\mu)$, the instant form hadronic spinors $u_\sigma(\underline{p}^\mu)$ and $v_\sigma(\underline{p}^\mu)$, $\sigma = j, j-1, \dots, -j$, were recently constructed explicitly (following Weinberg [1] and Ryder [9]) in Refs. [3(a-c), 3(g)]. A brief report, sufficient for the present discussion, can be found in Ref. [3(b)]. Here we only remark that the construction of instant form spinors follows the steps outlined in Eqs.(10-19) above, with the only difference that one starts with transformation Λ of Eq.(2) rather than L of Eq.(10).

The instant form hadronic spinors of Refs. [3(a-c), 3(g)] satisfy the following normalisation properties

$$\bar{u}_\sigma(\underline{p}^\mu) u_{\sigma'}(\underline{p}^\mu) = m^{2j} \delta_{\sigma\sigma'} \quad (29)$$

$$\bar{v}_\sigma(\underline{p}^\mu) v_{\sigma'}(\underline{p}^\mu) = -m^{2j} \delta_{\sigma\sigma'} \quad (30)$$

$$\bar{u}_\sigma(\underline{p}^\mu) v_{\sigma'}(\underline{p}^\mu) = 0 = \bar{v}_\sigma(\underline{p}^\mu) u_{\sigma'}(\underline{p}^\mu) . \quad (31)$$

¹ Note, however, a slightly different normalization and convention chosen by Lepage and Brodsky for the odd spinor parity spinors: $u_\uparrow = \sqrt{2}\mathcal{U}_{1/2}$, $u_\downarrow = \sqrt{2}\mathcal{U}_{-1/2}$, $v_\uparrow = \sqrt{2}\mathcal{V}_{-1/2}$, $v_\downarrow = -\sqrt{2}\mathcal{V}_{1/2}$.

where

$$\bar{u}_\sigma(\underline{p}^\mu) = [u_\sigma(\underline{p}^\mu)]^\dagger \gamma^\circ, \quad \bar{v}_\sigma(\underline{p}^\mu) = [v_\sigma(\underline{p}^\mu)]^\dagger \gamma^\circ \quad (32)$$

with γ° having the form identical to Γ° of Eq.(24).

In what follows we assume that p^μ and \underline{p}^μ correspond to the same physical momentum. The connection between the front form and instant form spinors is then established by noting that on general algebraic grounds we can express the instant form hadronic spinors as linear combination of the front form hadronic spinors. That is

$$u_\sigma(\underline{p}^\mu) = \Omega_{\sigma h}^{(u\mathcal{U})} \mathcal{U}_h(p^\mu) + \Omega_{\sigma h}^{(u\mathcal{V})} \mathcal{V}_h(p^\mu), \quad (33)$$

$$v_\sigma(\underline{p}^\mu) = \Omega_{\sigma h}^{(v\mathcal{U})} \mathcal{U}_h(p^\mu) + \Omega_{\sigma h}^{(v\mathcal{V})} \mathcal{V}_h(p^\mu), \quad (34)$$

where the sum on the repeated indices is implicit.

We now multiply Eq. (33) by $\bar{\mathcal{U}}_{h'}(p^\mu)$ from the left, and using the orthonormality relations, Eqs. (26-28), we get

$$\Omega_{\sigma h}^{(u\mathcal{U})} = \frac{1}{m^{2j}} [\bar{\mathcal{U}}_h(p^\mu) u_\sigma(\underline{p}^\mu)] . \quad (35)$$

Similarly by multiplying Eq. (34) from the left by $\bar{\mathcal{V}}_{h'}(p^\mu)$ and again using the orthonormality relations, Eqs. (26-28), we obtain

$$\Omega_{\sigma h}^{(v\mathcal{V})} = -\frac{1}{m^{2j}} [\bar{\mathcal{V}}_h(p^\mu) v_\sigma(\underline{p}^\mu)] . \quad (36)$$

Further it is readily verified, e.g. by using the results presented in Appendix B here and explicit expressions for $u_\sigma(\underline{p}^\mu)$ and $v_\sigma(\underline{p}^\mu)$ found in Refs. [3(a-c),3(g)], that

$$\bar{\mathcal{V}}_h(p^\mu) u_\sigma(\underline{p}^\mu) = 0 = \bar{\mathcal{U}}_h(p^\mu) v_\sigma(\underline{p}^\mu); \quad (37)$$

which yields

$$\Omega_{\sigma h}^{(u\mathcal{V})} = 0 = \Omega_{\sigma h}^{(v\mathcal{U})} . \quad (38)$$

Finally, we exploit the facts

$$\{\Gamma^5, \Gamma^\circ\} = 0, \quad \Gamma^{5\dagger} = \Gamma^5, \quad \text{and} \quad (\Gamma^5)^2 = I, \quad (39)$$

to conclude that $\bar{\mathcal{V}}_h(p^\mu) v_\sigma(\underline{p}^\mu) = -\bar{\mathcal{U}}_h(p^\mu) u_\sigma(\underline{p}^\mu)$. Thus the matrix which connects the instant form spinors with front form spinors reads

$$\Omega(j) = \begin{bmatrix} B(j) & 0 \\ 0 & B(j) \end{bmatrix}, \quad (40)$$

where $B(j)$ is a $(2j+1) \times (2j+1)$ matrix with elements $B_{\sigma h} = \bar{\mathcal{U}}_h(p^\mu) u_\sigma(\underline{p}^\mu) = \Omega_{\sigma h}^{(u\mathcal{U})} = \Omega_{\sigma h}^{(v\mathcal{V})}$.

The explicit expressions for $\Omega(j)$ are presented in Appendix C. For spin one half the transformation matrix $\Omega(1/2)$ computed by us coincides with the celebrated ‘‘Melosh transformation’’ given by Melosh in Ref. [5, Eq. 26] and by Dziembowski in Ref. [6, Eq. A8]. As formally demonstrated by Kondratyuk and Terent’ev [15], the transformation matrix $\Omega(j)$ represents a pure rotation of the spin basis. However, since $\Omega(j)$ has block zeros off-diagonal, what manifestly emerges here is that this rotation does not mix the even and odd spinor parity spinors.

V. SUMMARY

Within the framework of Weinberg-Soper formalism [1,2] we constructed explicit hadronic spinors for arbitrary spin in the front form, and established their connection with the instant form fields. For a given a spin, j , there are $(2j + 1)$ hadronic spinors with *even* spinor parity, $\mathcal{U}_h(p^\mu)$; and $(2j + 1)$ hadronic spinors with *odd* spinor parity, $\mathcal{V}_h(p^\mu)$. The normalisation of these spinors is so chosen that for the *massless* particles the $\mathcal{U}_h(\hat{p}^\mu)$ and $\mathcal{V}_h(\hat{p}^\mu)$ identically vanish; and only $\mathcal{U}_{h=\pm j}(p^\mu)$ and $\mathcal{V}_{h=\pm j}(p^\mu)$ survive. The simplest choice of this normalisation is given by Eq. (23). Next we constructed the matrix $\Omega(j)$ which provides the connection between the front form hadronic spinors with the more familiar [i.e. more “familiar” at least for spin one half case] instant form hadronic spinors. We verified that the transformation matrix $\Omega(1/2)$ coincides with the well known “Melosh transformation” [5,6], and the spin one half spinors are in agreement with the previous results of Lepage and Brodsky [4]. Explicit results for $\mathcal{U}_h(p^\mu)$, $\mathcal{V}_h(p^\mu)$ and $\Omega(j)$ up to spin two are found in Appendixes B and C here.

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Expansions for the $\exp(\eta \hat{b} \cdot \vec{J})$ and $\exp(-\eta \hat{b}^* \cdot \vec{J})$ up to Spin Two This appendix provides the expansions for the $\exp(\eta \hat{b} \cdot \vec{J})$ and $\exp(-\eta \hat{b}^* \cdot \vec{J})$, up to $j = 2$, which appear in the spinor-boost matrix $M(L)$, Eq. (19).

$$j = \frac{1}{2}$$

$$\exp(\eta \hat{b} \cdot \vec{J}) = \cosh(\eta/2) I + (\hat{b} \cdot \vec{\sigma}) \sinh(\eta/2), \quad (41)$$

$$\exp(-\eta \hat{b}^* \cdot \vec{J}) = \cosh(\eta/2) I - (\hat{b}^* \cdot \vec{\sigma}) \sinh(\eta/2). \quad (42)$$

$$j = 1$$

$$\exp(\eta \hat{b} \cdot \vec{J}) = I + 2(\hat{b} \cdot \vec{J})^2 \sinh^2(\eta/2) + 2(\hat{b} \cdot \vec{J}) \cosh(\eta/2) \sinh(\eta/2), \quad (43)$$

$$\exp(-\eta \hat{b}^* \cdot \vec{J}) = I + 2(\hat{b}^* \cdot \vec{J})^2 \sinh^2(\eta/2) - 2(\hat{b}^* \cdot \vec{J}) \cosh(\eta/2) \sinh(\eta/2). \quad (44)$$

$$j = \frac{3}{2}$$

$$\begin{aligned} \exp(\eta \hat{b} \cdot \vec{J}) = & \cosh(\eta/2) \left[I + \frac{1}{2} \left\{ (2\hat{b} \cdot \vec{J})^2 - I \right\} \sinh^2(\eta/2) \right] \\ & + (2\hat{b} \cdot \vec{J}) \sinh(\eta/2) \left[I + \frac{1}{6} \left\{ (2\hat{b} \cdot \vec{J})^2 - I \right\} \sinh^2(\eta/2) \right], \end{aligned} \quad (45)$$

$$\begin{aligned} \exp\left(-\eta \hat{b}^* \cdot \vec{J}\right) &= \cosh(\eta/2) \left[I + \frac{1}{2} \left\{ \left(2\hat{b}^* \cdot \vec{J}\right)^2 - I \right\} \sinh^2(\eta/2) \right] \\ &\quad - \left(2\hat{b}^* \cdot \vec{J}\right) \sinh(\eta/2) \left[I + \frac{1}{6} \left\{ \left(2\hat{b}^* \cdot \vec{J}\right)^2 - I \right\} \sinh^2(\eta/2) \right]. \end{aligned} \quad (46)$$

$j = 2$

$$\begin{aligned} \exp\left(\eta \hat{b} \cdot \vec{J}\right) &= I + 2 \left(\hat{b} \cdot \vec{J}\right)^2 \sinh^2(\eta/2) + \frac{2}{3} \left(\hat{b} \cdot \vec{J}\right)^2 \left\{ \left(\hat{b} \cdot \vec{J}\right)^2 - I \right\} \sinh^4(\eta/2) \\ &\quad + 2 \left(\hat{b} \cdot \vec{J}\right) \cosh(\eta/2) \sinh(\eta/2) + \frac{4}{3} \left(\hat{b} \cdot \vec{J}\right) \left\{ \left(\hat{b} \cdot \vec{J}\right)^2 - I \right\} \cosh(\eta/2) \sinh^3(\eta/2), \end{aligned} \quad (47)$$

$$\begin{aligned} \exp\left(-\eta \hat{b}^* \cdot \vec{J}\right) &= I + 2 \left(\hat{b}^* \cdot \vec{J}\right)^2 \sinh^2(\eta/2) + \frac{2}{3} \left(\hat{b}^* \cdot \vec{J}\right)^2 \left\{ \left(\hat{b}^* \cdot \vec{J}\right)^2 - I \right\} \sinh^4(\eta/2) \\ &\quad - 2 \left(\hat{b}^* \cdot \vec{J}\right) \cosh(\eta/2) \sinh(\eta/2) - \frac{4}{3} \left(\hat{b}^* \cdot \vec{J}\right) \left\{ \left(\hat{b}^* \cdot \vec{J}\right)^2 - I \right\} \cosh(\eta/2) \sinh^3(\eta/2). \end{aligned} \quad (48)$$

In Eqs. (41,42) the $\vec{\sigma}$ are the standard [9] Pauli matrices. In this appendix, I are the $(2j+1) \times (2j+1)$ identity matrices.

Front Form Hadronic Spinors up to Spin two We begin with collecting together front form hadronic spinors up to spin 2 *for even spinor parity*, first. In what follows we use the notation $p_r = p_x + i p_y$ and $p_\ell = p_x - i p_y$, c.f. Eq.(15),

Spin one half hadronic spinors with even spinor parity:

$$\mathcal{U}_{+\frac{1}{2}}(p^\mu) = \frac{1}{2} \sqrt{\frac{1}{p^+}} \begin{bmatrix} p^+ + m \\ p_r \\ p^+ - m \\ p_r \end{bmatrix}, \quad \mathcal{U}_{-\frac{1}{2}}(p^\mu) = \frac{1}{2} \sqrt{\frac{1}{p^+}} \begin{bmatrix} -p_\ell \\ p^+ + m \\ p_\ell \\ -p^+ + m \end{bmatrix}. \quad (49)$$

Spin one hadronic spinors with even spinor parity:

$$\mathcal{U}_{+1}(p^\mu) = \frac{1}{2} \begin{bmatrix} p^+ + (m^2/p^+) \\ \sqrt{2} p_r \\ p_r^2/p^+ \\ p^+ - (m^2/p^+) \\ \sqrt{2} p_r \\ p_r^2/p^+ \end{bmatrix}, \quad \mathcal{U}_0(p^\mu) = m \sqrt{\frac{1}{2}} \begin{bmatrix} -p_\ell/p^+ \\ \sqrt{2} \\ p_r/p^+ \\ p_\ell/p^+ \\ 0 \\ p_r/p^+ \end{bmatrix}, \quad \mathcal{U}_{-1}(p^\mu) = \frac{1}{2} \begin{bmatrix} p_\ell^2/p^+ \\ -\sqrt{2} p_\ell \\ p^+ + (m^2/p^+) \\ -p_\ell^2/p^+ \\ \sqrt{2} p_\ell \\ -p^+ + (m^2/p^+) \end{bmatrix}. \quad (50)$$

Spin three half hadronic spinors with even spinor parity:

$$\mathcal{U}_{+\frac{3}{2}}(p^\mu) = \frac{1}{2} \sqrt{\frac{1}{p^+}} \begin{bmatrix} p^{+2} + (m^3/p^+) \\ \sqrt{3} p_r p^+ \\ \sqrt{3} p_r^2 \\ p_r^3/p^+ \\ p^{+2} - (m^3/p^+) \\ \sqrt{3} p_r p^+ \\ \sqrt{3} p_r^2 \\ p_r^3/p^+ \end{bmatrix}, \quad \mathcal{U}_{+\frac{1}{2}}(p^\mu) = \frac{m}{2} \sqrt{\frac{1}{p^+}} \begin{bmatrix} -\sqrt{3} m p_\ell/p^+ \\ p^+ + m \\ 2 p_r \\ \sqrt{3} p_r^2/p^+ \\ \sqrt{3} m p_\ell/p^+ \\ p^+ - m \\ 2 p_r \\ \sqrt{3} p_r^2/p^+ \end{bmatrix}, \quad (51)$$

$$\mathcal{U}_{-\frac{1}{2}}(p^\mu) = \frac{m}{2} \sqrt{\frac{1}{p^+}} \begin{bmatrix} \sqrt{3} p_\ell^2/p^+ \\ -2 p_\ell \\ p^+ + m \\ \sqrt{3} m p_r/p^+ \\ -\sqrt{3} p_\ell^2/p^+ \\ 2 p_\ell \\ -p^+ + m \\ \sqrt{3} m p_r/p^+ \end{bmatrix}, \quad \mathcal{U}_{-\frac{3}{2}}(p^\mu) = \frac{1}{2} \sqrt{\frac{1}{p^+}} \begin{bmatrix} -p_\ell^3/p^+ \\ \sqrt{3} p_\ell^2 \\ -\sqrt{3} p_\ell p^+ \\ p^{+2} + (m^3/p^+) \\ p_\ell^3/p^+ \\ -\sqrt{3} p_\ell^2 \\ \sqrt{3} p_\ell p^+ \\ -p^{+2} + (m^3/p^+) \end{bmatrix}. \quad (52)$$

Spin two hadronic spinors with even spinor parity:

$$\mathcal{U}_{+2}(p^\mu) = \frac{1}{2} \begin{bmatrix} p^{+2} + (m^4/p^{+2}) \\ 2p_r p^+ \\ \sqrt{6} p_r^2 \\ 2p_r^3/p^+ \\ p_r^4/p^{+2} \\ p^{+2} - (m^4/p^{+2}) \\ 2p_r p^+ \\ \sqrt{6} p_r^2 \\ 2p_r^3/p^+ \\ p_r^4/p^{+2} \end{bmatrix}, \quad \mathcal{U}_{+1}(p^\mu) = \frac{m}{2} \begin{bmatrix} -2m^2 p_\ell/p^{+2} \\ p^+ + (m^2/p^+) \\ \sqrt{6} p_r \\ 3p_r^2/p^+ \\ 2p_r^3/p^{+2} \\ 2m^2 p_\ell/p^{+2} \\ p^+ - (m^2/p^+) \\ \sqrt{6} p_r \\ 3p_r^2/p^+ \\ 2p_r^3/p^{+2} \end{bmatrix}, \quad (53)$$

$$\mathcal{U}_0(p^\mu) = \frac{m^2}{2} \begin{bmatrix} \sqrt{6} p_\ell^2/p^{+2} \\ -\sqrt{6} p_\ell/p^+ \\ 2 \\ \sqrt{6} p_r/p^+ \\ \sqrt{6} p_r^2/p^{+2} \\ -\sqrt{6} p_\ell^2/p^{+2} \\ \sqrt{6} p_\ell/p^+ \\ 0 \\ \sqrt{6} p_r/p^+ \\ \sqrt{6} p_r^2/p^{+2} \end{bmatrix}, \quad \mathcal{U}_{-1}(p^\mu) = \frac{m}{2} \begin{bmatrix} -2p_\ell^3/p^{+2} \\ 3p_\ell^2/p^+ \\ -\sqrt{6} p_\ell \\ p^+ + (m^2/p^+) \\ 2m^2 p_r/p^{+2} \\ 2p_\ell^3/p^{+2} \\ -3p_\ell^2/p^+ \\ \sqrt{6} p_\ell \\ -p^+ + (m^2/p^+) \\ 2m^2 p_r/p^{+2} \end{bmatrix}, \quad (54)$$

$$\mathcal{U}_{-2}(p^\mu) = \frac{1}{2} \begin{bmatrix} p_\ell^4/p^{+2} \\ -2p_\ell^3/p^+ \\ \sqrt{6}p_\ell^2 \\ -2p_\ell p^+ \\ p^{+2} + (m^4/p^{+2}) \\ -p_\ell^4/p^{+2} \\ 2p_\ell^3/p^+ \\ -\sqrt{6}p_\ell^2 \\ 2p_\ell p^+ \\ -p^{+2} + (m^4/p^{+2}) \end{bmatrix}. \quad (55)$$

An examination of the spinor-boost matrix $M(L)$, Eq.(19), implies that the *odd* spinor parity hadronic spinor can be obtained from the hadronic spinor of the *even* spinor parity via the following simple relation

$$\mathcal{V}_h(p^\mu) = \Gamma^5 \mathcal{U}_h(p^\mu), \quad (56)$$

where the matrix Γ^5 defined in Eq.(24) interchanges the top $(2j + 1)$ elements with the bottom $(2j + 1)$ elements of the hadronic spinors.

The Explicit Expressions for $\Omega(j)$, the Generalised Melosh Transformation, up to Spin Two

In this appendix we present explicit expressions for the matrix $\Omega(j)$, Eq. (40), which connects the front form hadronic spinors with the instant form hadronic spinors via Eqs. (33,34). As in Appendix B, $p_r = p_x + i p_y$ and $p_\ell = p_x - i p_y$, in what follows.

For spin one half, the matrix connecting the instant form hadronic spinors with the front form spinors is

$$\Omega(1/2) = \frac{1}{[2(E + m)p^+]^{1/2}} \begin{bmatrix} \beta(1/2) & 0 \\ 0 & \beta(1/2) \end{bmatrix}, \quad (57)$$

where the 2×2 block matrix $\beta(1/2)$ is defined as

$$\beta(1/2) = \begin{bmatrix} p^+ + m & -p_r \\ p_\ell & p^+ + m \end{bmatrix}. \quad (58)$$

For spin one, the matrix connecting the instant form hadronic spinors with the front form spinors is

$$\Omega(1) = \frac{1}{[2(E + m)p^+]} \begin{bmatrix} \beta(1) & 0 \\ 0 & \beta(1) \end{bmatrix}, \quad (59)$$

where the 3×3 block matrix $\beta(1)$ is defined as

$$\beta(1) = \begin{bmatrix} (p^+ + m)^2 & -\sqrt{2} (p^+ + m) p_r & p_r^2 \\ \sqrt{2} (p^+ + m) p_\ell & 2 [(E + m) p^+ - p_r p_\ell] & -\sqrt{2} (p^+ + m) p_r \\ p_\ell^2 & \sqrt{2} (p^+ + m) p_\ell & (p^+ + m)^2 \end{bmatrix}. \quad (60)$$

For spin three half, the matrix connecting the instant form hadronic spinors with the front form spinors is

$$\Omega(3/2) = \frac{1}{[2 (E + m) p^+]^{3/2}} \begin{bmatrix} \beta(3/2) & 0 \\ 0 & \beta(3/2) \end{bmatrix}, \quad (61)$$

where the 4×4 block matrix $\beta(3/2)$ is defined as

$$\beta(3/2) = \begin{bmatrix} (p^+ + m)^3 & -\sqrt{3} (p^+ + m)^2 p_r & \sqrt{3} (p^+ + m) p_r^2 & -p_r^3 \\ \sqrt{3} (p^+ + m)^2 p_\ell & [(p^+ + m)^2 - 2p_r p_\ell] (p^+ + m) & -[2(p^+ + m)^2 - p_r p_\ell] p_r & \sqrt{3} (p^+ + m) p_r^2 \\ \sqrt{3} (p^+ + m) p_\ell^2 & [2(p^+ + m)^2 - p_r p_\ell] p_\ell & [(p^+ + m)^2 - 2p_r p_\ell] (p^+ + m) & -\sqrt{3} (p^+ + m)^2 p_r \\ p_\ell^3 & \sqrt{3} (p^+ + m) p_\ell^2 & \sqrt{3} (p^+ + m)^2 p_\ell & (p^+ + m)^3 \end{bmatrix}. \quad (62)$$

For spin two, the matrix connecting the instant form hadronic spinors with the front form spinors is

$$\Omega(2) = \frac{1}{[2(E + m)p^+]^2} \begin{bmatrix} \beta(2) & 0 \\ 0 & \beta(2) \end{bmatrix}, \quad (63)$$

where the 5×5 block matrix $\beta(2)$ is defined via the following five columns

$$\begin{aligned} \beta(2)_{\alpha,1} &= \begin{bmatrix} (p^+ + m)^4 \\ 2(p^+ + m)^3 p_\ell \\ \sqrt{6} (p^+ + m)^2 p_\ell^2 \\ 2(p^+ + m) p_\ell^3 \\ p_\ell^4 \end{bmatrix}, \quad \beta(2)_{\alpha,2} = \begin{bmatrix} -2(p^+ + m)^3 p_r \\ 2[(E + m)p^+ - 2p_r p_\ell] (p^+ + m)^2 \\ \sqrt{6} [(E + m)p^+ - p_r p_\ell] (p^+ + m) p_\ell \\ [6p^+(E + m) - 4p_r p_\ell] p_\ell^2 \\ 2(p^+ + m) p_\ell^3 \end{bmatrix}, \\ \beta(2)_{\alpha,3} &= \begin{bmatrix} \sqrt{6} (p^+ + m)^2 p_r^2 \\ \sqrt{6} [-(E + m)p^+ + p_r p_\ell] (p^+ + m) p_r \\ 2[2p^{+2}(E + m)^2 - 3(p^+ + m)^2 p_r p_\ell] \\ \sqrt{6} [(E + m)p^+ - p_r p_\ell] (p^+ + m) p_\ell \\ \sqrt{6} (p^+ + m)^2 p_\ell^2 \end{bmatrix}, \\ \beta(2)_{\alpha,4} &= \begin{bmatrix} -2(p^+ + m) p_r^3 \\ [6p^+(E + m) - 4p_r p_\ell] p_r^2 \\ \sqrt{6} [-(E + m)p^+ + p_r p_\ell] (p^+ + m) p_r \\ 2[(E + m)p^+ - 2p_r p_\ell] (p^+ + m)^2 \\ 2(p^+ + m)^3 p_\ell \end{bmatrix}, \quad \beta(2)_{\alpha,5} = \begin{bmatrix} p_r^4 \\ -2(p^+ + m) p_r^3 \\ \sqrt{6} (p^+ + m)^2 p_r^2 \\ -2(p^+ + m)^3 p_r \\ (p^+ + m)^4 \end{bmatrix}. \end{aligned} \quad (64)$$

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TABLES

TABLE I. Algebra associated with the stability group of the $x^+ = 0$ plane. The Commutator [Element in the *first* column, Element in the *first* row] = The element at the intersetction of *the* row and column.

	P_1	P_2	J_3	K_3	P_-	G_1	G_2
P_1	0	0	$-iP_2$	0	0	iP_-	0
P_2	0	0	iP_1	0	0	0	iP_-
J_3	iP_2	$-iP_1$	0	0	0	iG_2	$-iG_1$
K_3	0	0	0	0	iP_-	iG_1	iG_2
P_-	0	0	0	$-iP_-$	0	0	0
G_1	$-iP_-$	0	$-iG_2$	$-iG_1$	0	0	0
G_2	0	$-iP_-$	iG_1	$-iG_2$	0	0	0